

Absence of Boolean Percolation on Doubling Graphs

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October 21, 2014

Abstract

We consider the Boolean discrete percolation model on graphs satisfying a doubling metric condition. We study sufficient conditions on the distribution of the radii of balls placed at the points of a Bernoulli point process for the absence of percolation, provided that the retention parameter of the underlying point process is small enough. We exhibit three families of interesting graphs where the main result of this work holds. An interesting example of such graphs is given by the Cayley graph of the discrete Heisenberg group which can not be embedded in \mathbb{R}^n for any n . Therefore, the absence of percolation in doubling graphs does not follow from the subcriticality of the Boolean percolation model in \mathbb{R}^n and standard coupling arguments. Finally, we give sufficient conditions for ergodicity of the Boolean discrete percolation model.

Introduction

The aim of this work is to study sufficient conditions for subcriticality in the discrete Boolean model of percolation in general graphs. We give now an informal description of the Boolean model for percolation. Consider a simple point process \mathcal{X} in some polish, locally compact metric space (Γ, d) . Then, at each point of \mathcal{X} , center a ball of random radius. Assume that the radius are independent, identically distributed and independent of \mathcal{X} . Thus, Γ is partitioned in two regions, the occupied region, which is defined as the union of all random balls, and the vacant region, which is the complement of the occupied region.

In this paper we consider the case in which Γ is a doubling weighted graph equipped with the weighted graph distance and the underlying point process \mathcal{X} is a Bernoulli point process with retention parameter p for some $p \in (0, 1)$. In this setting, we prove the absence of unbounded connected components on the occupied region.

This model is the discrete counterpart of the Poisson Boolean model of continuum percolation. In the Poisson Boolean model a ball of random radius is centered at each point of a

homogeneous Poisson point process with density λ on \mathbb{R}^n . The corresponding radii form an independent and identically distributed collection of non-negative random variables which are also independent of the point process. Denote by \mathcal{B} the union of these balls and by \mathcal{C} the connected component of \mathcal{B} containing the origin. Let R be one of the random radii and denote by \mathbf{P} the law governing the continuous boolean model. Also, denote by \mathbf{E} the corresponding expectation operator. In [9], Hall proved that for values of λ small enough, \mathcal{C} is almost surely bounded provided that $\mathbf{E}[R^{2n-1}]$ is finite. In [14], Meester and Roy proved that if $n \geq 2$, then the expected number of balls in the occupied component which contains the origin is finite whenever λ is small enough if, and only if, $\mathbf{E}[R^{2n}]$ is finite. Also, they proved that if $\mathbf{E}[R^{2n-1}]$ is finite then $\mathbf{P}(\text{number of balls in any occupied component is finite}) = 1$ provided that λ is small enough. In [6], Gouere showed that the set \mathcal{C} is almost surely bounded for small enough λ if and only if $\mathbf{E}[R^n]$ is finite.

The Boolean model of percolation on \mathbb{R}^n belongs to the family of continuum percolation models. In fact, the history of continuum percolation began in 1961 when W. Gilbert [5] introduced the random connection model on the plane. In 1985, S. Zuev and A. Sidorenko [21] considered continuum models of percolation where points are chosen randomly in space and surrounded by shapes which can be random or fixed. In that work the authors studied the relation between critical parameters associated to that model. For a comprehensive study of continuum models of percolation, see the book of R. Meester and R. Roy [14].

In 2001, I. Benjamini and O. Schram considered different percolation models in the hyperbolic plane and on regular tilings in the hyperbolic plane. Recently, the Boolean model of percolation has received considerable attention. In [19], J. Tykesson studied the Poisson Boolean continuum model for percolation in \mathbb{H}^n . He showed that, for certain values of the intensity of the underlying Poisson point process, there are infinitely many unbounded components in the occupied and vacant regions. In 2014, C. Coletti and S. Grynberg [3] used the existence of a subcritical phase of the discrete version of this model on the n -dimensional integer lattice \mathbb{Z}^n to construct, forward in time, interacting particle systems with generator admitting a Kalikow-type decomposition.

We finish this introduction by pointing out that the discrete Boolean model of percolation in graphs where the underlying point process is a Bernoulli point process and the balls under consideration are (closed) balls of radius $1/2$ (any positive number lower than 1 would work as well) corresponds to the case of independent site percolation. In fact, the theory of discrete percolation began before the theory of continuum percolation with the paper of S. Broadbent and J. Hammersley in 1957 where they introduced iid bond percolation. Their motivation was to understand the flow of a liquid through a porous media. For a long time, percolation on the n -dimensional integer lattice concentrated the attention of the probabilistic and physical community. In 1989, percolation theory underwent a remarkable change with the work of R. Lyons. He studied percolation on regular tree or tree-like graphs. In the nineties appeared the first results on percolation on graphs beyond \mathbb{Z}^n and tree-like graphs. In 1996, I. Benjamini and O. Schramm proposed a comprehensive study of percolation on Cayley graphs. O. Haggstrom [8] and a. Procacci [16] proposed to study percolation problems on general graphs.

In this paper we prove that if the underlying graph satisfies a doubling condition and if

the family of random radii are i.i.d. random variables with finite $\dim_A(\Gamma)$ -moment, where $\dim_A(\Gamma)$ is the corresponding Assouad dimension which will be defined in section 1, then the connected components arising in the discrete Boolean model are almost surely finite for sufficiently small values of p . We also prove that such behavior does not occur if the random radii have infinite $\dim_A(\Gamma)$ -moment.

This paper is organized as follows: In section 1 we describe the Boolean discrete percolation model and state the main result of this work, Theorem 1, which says about the absence of percolation on doubling graphs. See Theorem 1 below for a precise statement of this result. This result is proved in section 2. In subsection 1.5 we also provide some interesting examples of graphs satisfying the assumptions of Theorem 1. Indeed, Theorem 1 is valid for the family of graph with polynomial growth. In particular, Theorem 1 holds for lattices and nilpotent groups. Then, and as expected, we prove that the whole space is covered if the growth function has infinite moment. Finally, in section 3, we address the problem of determining sufficient conditions for ergodicity of the Boolean discrete percolation model.

1 Definitions, notation and statement of the main result

1.1 Doubling Graphs

Throughout this paper, \mathbb{N}_0 will denote the set of non-negative integer numbers and $\Gamma = (V, E)$ will denote a countable infinite connected weighted graph. We regard Γ as a metric space with the metric d given by the weighted distance on the graph Γ . We write $|\cdot|$ for the counting measure on V . Thus, for any finite set $A \subset V$, $|A|$ denotes the number of elements of A . Also, $B(v, r) = \{u \in V : d(u, v) \leq r\}$ denotes the closed ball of radius r centered at v and $S(v, r) = \{u \in V : d(u, v) = r\}$ denotes the sphere of radius r in Γ around v .

We say that (Γ, d) is a *doubling metric graph* if there exists a non-negative constant C so that any ball B in Γ can be covered with at most C balls whose radius is half the radius of B . For further reading on doubling metric spaces, see [11] and [13]. Doubling graphs arise naturally in many applications. For instance, metric embedding of doubling graphs turned out to be useful for algorithm design. The doubling condition has also appeared in the study of the problem of designing routing algorithms for networks with structure parametrized by its doubling dimension, see e.g. [2] and [17]. A related but stronger condition is the volume and time doubling assumption on graphs which has been used to prove upper and lower off-diagonal, sub-Gaussian transition probability estimates for strongly recurrent random walks, see [18].

Assouad Dimension Now we introduce the concept of Assouad dimension which will be used to state our main result. To begin with, let $\epsilon > 0$ be given. We call a subset $A \subset V$ ϵ -*separated* if $d(v, w) \geq \epsilon$ for all distinct $v, w \in A$. Let $N(B, \epsilon)$ be the maximal cardinality

of an ϵ -separated subset of B . Then (Γ, d) is doubling if and only if there exists $C < \infty$ such that

$$N(B(v, r), r/2) \leq C \quad (1)$$

for all balls $B(v, r)$ in Γ . An easy inductive argument let us show that if (1) holds, then there exists C' and β depending only on C such that

$$N(B(v, r), \epsilon r) \leq C' \epsilon^{-\beta} \quad (2)$$

for any ball $B(v, r)$ in Γ and any $0 < \epsilon < 1$. For instance, we may take $\beta = \log_2 C$. Thus, the doubling condition is a finite-dimensional hypothesis which controls the growth of the cardinalities of separated subsets of any ball at any scale and location. Finally, we define the *Assouad dimension* as the infimum of all $\beta > 0$ such that (2) holds. Denote the Assouad dimension of Γ by $\dim_A(\Gamma)$. A useful fact about the Assouad dimension which follows directly from its definition and which will be used later is that it is possible to control the volume of any ball in terms of $\dim_A(\Gamma)$. It follows from (2), by taking $\epsilon = 1/r$, that there exists a constant C_1 depending only on $\dim_A(\Gamma)$ such that

$$|B(v, r)| \leq C_1 r^{\dim_A(\Gamma)} \quad (3)$$

for any $v \in V$ and any $r \in \mathbb{N}$.

1.2 Marked Point Process

A *Bernoulli point process* on Γ with retention parameter $p \in (0, 1)$, is a family of independent $\{0, 1\}$ -valued random variables $\mathcal{X} = (X_v : v \in V)$ such that $\mathbf{P}(X_v = 1) = p$. Identify the family of random variables \mathcal{X} with the random subset \mathcal{P} of V defined by $\mathcal{P} = \{v \in V : X_v = 1\}$ whose distribution is a product measure whose marginals at each vertex v are Bernoulli distribution of parameter p .

By a *Bernoulli marked point process* on Γ we mean a pair $(\mathcal{X}, \mathcal{R})$ formed by a Bernoulli point process \mathcal{X} on Γ and a family of independent \mathbb{N}_0 -valued random variables $\mathcal{R} = (R_v : v \in V)$ called marks. We assume that these marks are independent of the point process \mathcal{X} .

Let $(\mathcal{X}, \mathcal{R})$ be a marked point process on Γ with retention parameter p and marks distributed according to the probability function ν . We denote by $\mathbf{P}_{p, \nu}$ and $\mathbf{E}_{p, \nu}$ respectively the probability measure and the expectation operator induced by $(\mathcal{X}, \mathcal{R})$.

1.3 Random Graphs and Percolation.

Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process on Γ . Then we define an associated random graph $\mathcal{G}(\mathcal{X}, \mathcal{R}) = (V, \mathcal{E})$ as the undirected random graph with vertex set V and edge set \mathcal{E} defined by the condition $\{v, w\} \in \mathcal{E}$ if, and only if, $X_v = 1$ and $w \in B(v, R_v)$ or $X_w = 1$ and $v \in B(w, R_w)$.

A *path* on $\mathcal{G}(\mathcal{X}, \mathcal{R})$ is a sequence of distinct vertex v_0, v_1, \dots, v_n with $v_{i-1} \neq v_i$ such that $\{v_{i-1}, v_i\} \in \mathcal{E}$, $i = 1, \dots, n$.

A set of vertex $C \subset V$ is connected if, for any pair of distinct vertex v and w in C , there exists a path on $\mathcal{G}(\mathcal{X}, \mathcal{R})$ using vertices only from C , starting at v and ending at w . The connected components of the graph $\mathcal{G}(\mathcal{X}, \mathcal{R})$ are its maximal connected subgraphs.

The cluster $C(v)$ of vertex v is the connected component of the graph $\mathcal{G}(\mathcal{X}, \mathcal{R})$ containing v . Define the *Percolation event* as follows:

$$[\text{Percolation}] := \bigcup_{v \in V} \{|C(v)| = \infty\}. \quad (4)$$

1.4 Main Result

Now we state the main result of this work

Theorem 1 *Let Γ be a doubling graph. Let $(\mathcal{X}, \mathcal{R})$ be a marked point process on Γ with retention parameter p and marks distributed according to the probability function ν . Let R be a random variable whose law is ν . If*

$$\mathbf{E}_\nu[R^{\dim_A(\Gamma)}] < \infty, \quad (5)$$

then there exists $p_0 > 0$ such that $\mathbf{P}_{p, \nu}(\text{Percolation}) = 0$ for all $p \leq p_0$.

Phase transition. Consider the Bernoulli Boolean discrete percolation model introduced above. Then replace the random radii in this model by the deterministic radius 0. What we get is the independent site percolation model in Γ . As in other percolation models we define the percolation probability $\theta(p)$ by $\theta(p) = \mathbf{P}[\text{Percolation}]$. A simple coupling arguments gives the monotonicity of $\theta(p)$ in p . Thus, the critical parameter

$$p_c(\Gamma) = \sup\{p : \theta(p) = 0\} \quad (6)$$

is well defined. Then, a direct coupling with site percolation in Γ yields $p_c(\Gamma) \leq p_c^s(\Gamma)$, where $p_c^s(\Gamma)$ is the critical parameter for independent site percolation in Γ .

Theorem 2 *If Γ is the Cayley graph of a group G with at most polynomial growth containing a subgroup isomorphic to \mathbb{Z}^2 , then $p_c < 1$.*

The proof of the previous theorem follows from the relation stated above between the critical parameters for independent site and Boolean percolation and Theorem 7.17 and Corollary 7.18 in [12] which state the analogous result to that stated in Theorem 2 above for independent site percolation in graphs.

Complete Coverage. We complement the result of Theorem 1 by establishing a sufficient condition for complete coverage of V . For any $A \subset V$, define $\Lambda(A) = \bigcup_{v \in A \cap \mathcal{P}} B(v, R_v)$.

Theorem 3 *Let $(\mathcal{X}, \mathcal{R})$ be a marked point process on Γ with retention parameter p and marks distributed according to the probability function ν . Let R be a random variable whose law is ν . If there exists $v \in V$ such that $\mathbf{E}_\nu[R^{\dim_A(\Gamma)}] = \infty$, then for any $p \in (0, 1]$, $\Lambda(V) = V$.*

1.5 Examples of Doubling Graphs

The first and fundamental example of a doubling metric graph is \mathbb{Z}^n which has polynomial growth of order n . This example may be generalized, at least, in three ways: graphs with polynomial growth, Cayley graphs of nilpotent groups and lattices.

We observe that an easy way to obtain other examples of doubling graphs is to consider subgraphs of a doubling graph, since the property of a graph being doubling is hereditary, i.e., a subspace of a doubling metric space is doubling ([11], B.2.5).

Graphs with Polynomial Growth An embracing family of doubling metric graphs is the one composed of graphs with polynomial growth. For a comprehensive study of such graphs see [10] and [7]. For each $v \in V$, the *growth function* $\gamma(v, \cdot) : \mathbb{N} \rightarrow \mathbb{N}_0$ with respect to the vertex v , is given by

$$\gamma(v, r) := |B(v, r)|. \quad (7)$$

It is worth mentioning that, in the case of transitive graphs the growth function does not depend on the choice of a particular vertex v and in the case of non-transitive graphs, the growth functions for two different vertexes differs only by a constant.

We assume that for each $r \in \mathbb{N}$ $\gamma(r) = \sup_{v \in V} \gamma(v, r) < \infty$. Also, we say that a given graph has *polynomial growth* if there exist constants C and d such that its associated growth function satisfies the inequality $C^{-1}r^d \leq \gamma(v, r) \leq Cr^d$ for any $v \in V$ and $r > 0$.

A graph with polynomial growth satisfies a condition which is slightly stronger than being a doubling metric space. It may be proved that in this case the graph is a doubling measure space. To keep the paper self-contained we give the definition of doubling measure space.

Definition 4 *Let (M, d) be a metric space. A positive Borel measure space μ on M is said to be doubling if there exists a constant $C > 0$ such that*

$$\mu(2B) \leq C\mu(B)$$

for all balls B in M .

It follows from the previous definition that a graph with polynomial growth is a doubling measure space with respect to the counting measure. Since metric spaces admitting a doubling measure are doubling metric spaces ([7], B.3) we may conclude that graphs with polynomial growth form a family of doubling metric graphs.

Cayley Graphs of Nilpotent Groups Let G be a finitely generated group with generating set H . Assume that the identity element $e \notin H$. We may associate the Cayley graph $\Gamma(G, H)$ of G with respect to H , whose vertices are the elements of G . The set of edges $E(G, H)$ is defined as follows,

$$E(G, H) = \{(g, gh) | g \in G, h \in H \cup H^{-1}\}.$$

Then, define the growth of the group G as the growth of the Cayley graph $\Gamma(G, H)$ with respect to some (any) generating set H . The order of growth is well defined, since changing the generating set H only changes the constants appearing in the bounds of the growth function.

Theorem 5 [Gromov [7] and Wolf [20]] *A finitely generated group has polynomial growth if and only if it is almost nilpotent.*

It follows from Theorem 5 above that Cayley graphs of nilpotent groups have polynomial growth which in turns implies that they form a family of doubling metric graphs.

A concrete example of a group with polynomial growth is the discrete Heisenberg group $H_3(\mathbb{Z})$, where

$$H_3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}.$$

Since the discrete Heisenberg group is nilpotent, it has polynomial growth. For further details we refer the reader to [7].

Remark 1 *It is worth mentioning that, by a discrete version of Pansu Theorem [15], the discrete Heisenberg group is an example of a doubling metric space which cannot be embedded in \mathbb{R}^n , for any n . Therefore, the absence of percolation in doubling graphs does not follow from the subcriticality of the Boolean percolation model in \mathbb{R}^n and standard coupling arguments.*

Lattices Let $\{K_1, \dots, K_n\}$ be a family of compact and convex subsets of \mathbb{R}^n and let ϕ_i^j be a family of isometries from K_i to \mathbb{R}^n , $1 \leq i \leq n$ such that

$$\mathbb{R}^n = \bigcup_{i,j} \phi_i^j(K_i)$$

and

$$\text{int}(\phi_i^j(K_i)) \cap \text{int}(\phi_s^l(K_s)) = \emptyset.$$

Here $\text{int}(A)$ stands for the interior of the set A . We call the family $\mathcal{K} := \{\phi_i^j(K_i), \forall i, j\}$ a *lattice*. Now we associate a graph Γ to \mathcal{K} as follows. For each i , choose a point $p_i \in \text{int}(K_i)$ and let the vertexes of the graph be the points $\phi_i^j(p_i)$. We say that the pair $\{p_i^j, p_k^l\}$ is an edge if and only if $K_i^j \cap K_k^l \neq \emptyset$. Since the lattice just constructed is a subset of \mathbb{R}^n ,

it inherits the doubling property. An interesting family of lattices in \mathbb{R}^n is the family of quasi-aperiodic tilings of \mathbb{R}^n , in particular, the family of Penrose tilings.

The previous definition of lattices of \mathbb{R}^n can be easily generalized for lattices of a Lie group G , and in the case that G is nilpotent, then one can prove that this lattice is also a doubling metric space. For further details, see [1].

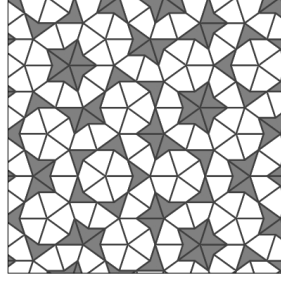


Figure 1: A Penrose Tiling.

2 Proof of Theorem 1

The proof of Theorem 1 will be divided into two steps. In the first step we introduce two families of events, $G(v, r)$ and $H(v, r)$, in order to study the diameter of the cluster $C(v)$. The family of events $G(v, r)$ is helpful to understand the behavior of the the diameter of the cluster $C(v)$ on the subgraph of $\mathcal{G}(\mathcal{X}, \mathcal{R})$ induced by the point process on $B(v, 10r)$. The family of events $H(v, r)$ provides a way to take care of the influence of the point process $(\mathcal{X}, \mathcal{R})$ from the exterior of the ball $B(v, r)$. Our aim in this step is to show that the probability of the percolation event can be controlled by the probabilities of the events $G(v, r)$. In the second step, we will show that if the radii are not too large, then the occurrence of the event $G(v, r_1)$ implies the occurrence of two independent events $G(u, r_2)$ and $G(w, r_2)$ where $r_1 = 10r_2$. Our aim in this step is to show that the probability of the events $G(v, r_1)$ can be bounded from above by the square of the probability of the events $G(u, r_2)$ plus a quantity that goes to zero when r_1 goes to infinity. This provides a way to take care of the probabilities of the events $G(v, r)$ that allows us to show that for p small enough, $\mathbf{P}_p(G(v, r))$ goes to zero when r goes to infinity.

2.1 Controlling the diameter of the clusters $C(v)$

For each $v \in V$, let $D_v = \inf\{r \geq 0 : C(v) \subset B(v, r)\}$. The percolation event is equivalent to the event $\bigcup_{v \in V} \{D_v = \infty\}$. The proof of Theorem 1 is reduced to show that there exists $p_0 > 0$ such that for each $v \in V$, $\lim_{r \rightarrow \infty} \mathbf{P}_{p, v}(D_v > r) = 0$ for all $p < p_0$.

For each $v \in V$ we define two families of events to study the diameter of the cluster $C(v)$.

The family of events $G(v, r)$. Let B be a subset of V . Denote by $\mathcal{G}[B]$ the subgraph of $\mathcal{G}(\mathcal{X}, \mathcal{R})$ induced by B . Let A be a non-empty subset of V contained in B and let $v \in A$. We say that v is *disconnected from the exterior of A inside B* if the connected component of $\mathcal{G}[B]$ containing v is contained in A . Now we introduce the events $G(v, r)$. Let $v \in V$ and let $r \in \mathbb{N}$, we say that $G(v, r)$ does not occur if v is disconnected from the exterior of $B(v, 8r)$ inside $B(v, 10r)$.

The family of events $H(v, r)$. For each $v \in V$ and $r \in \mathbb{N}$, we define

$$H(v, r) = \left\{ \exists w \in \mathcal{P} \cap B(v, 10r)^c : R_w > \frac{d(w, v)}{10} \right\}. \quad (8)$$

The relation between the diameter of the cluster at v and the families of events defined above is established in the following lemma.

Lemma 6 *The following inclusion holds for all $r \in \mathbb{N}$:*

$$G(v, r)^c \cap H(v, r)^c \subset \{D_v \leq 8r\}. \quad (9)$$

Proof of Lemma 6. If the event $H(v, r)$ does not occur, then there are no sites of the point process with distance to v greater than $10r$ connected to the ball $B(v, 9r)$. Indeed, assume that $H(v, r)$ does not occur. Then for every $w \in \mathcal{P} \cap B(v, 10r)^c$ we have $d(w, v) - R_w \geq \frac{9}{10}d(w, v) > 9r$. Using the triangle inequality it is easy to verify that $d(u, v) \geq d(w, v) - R_w > 9r$ for all $u \in B(w, R_w)$. If $G(v, r)$ does not occur, then v is isolated from the exterior of $B(v, 8r)$. If, in addition, the event $H(v, r)$ does not occur, then the balls $B(w, R_w)$, $w \in \mathcal{P} \cap B(v, 10r)^c$, do not connect the vertex v to the complement of $B(v, 8r)$. Thus $D_v \leq 8r$. \square

From (9) we get

$$\mathbf{P}_{p, \nu}(D_v > 8r) \leq \mathbf{P}_{p, \nu}(G(v, r)) + \mathbf{P}_{p, \nu}(H(v, r)). \quad (10)$$

Notice that $\lim_{r \rightarrow \infty} \mathbf{P}_{p, \nu}(H(v, r)) = 0$ for all $p \in (0, 1)$. This is obvious because $H(v, r+1) \subset H(v, r)$ for all $r \in \mathbb{N}$ and $\bigcap_{r \in \mathbb{N}} H(v, r) = \emptyset$.

2.2 Controlling the probabilities of the events $G(v, r)$

To take care of the probabilities $\mathbf{P}_{p, \nu}(G(v, r))$ we introduce another family of events.

The family of events $\tilde{H}(v, r)$. For each $v \in V$ and $r \in \mathbb{N}$, we define

$$\tilde{H}(v, r) = \{\exists w \in \mathcal{P} \cap B(v, 100r) : R_w \geq r\}. \quad (11)$$

Lemma 7 *The following inclusion holds for all $r \in \mathbb{N}$:*

$$G(v, 10r) \cap \tilde{H}(v, r)^c \subset \left(\bigcup_{u \in A(v, r, 10)} G(u, r) \right) \cap \left(\bigcup_{w \in A(v, r, 80)} G(w, r) \right), \quad (12)$$

where $A(v, r, m)$ is a r -separated subset of $S(v, mr)$ such that $|A(v, r, m)| = N(S(v, mr), r)$.

Proof of Lemma 7. Fix $r \in \mathbb{N}$. First, assume that the event $G(v, 10r)$ occurs but the event $\tilde{H}(v, r)$ does not occur. Since $G(v, 10r)$ occurs we can go from the vertex v to the complement of the ball $B(v, 80r)$ just using balls $B(u, R_u)$ centered at points from $\mathcal{P} \cap B(v, 100r)$. In this way, we can go from the sphere $S(v, 10r)$ to the sphere $S(v, 80r)$. One of this balls, let say $B(u_*, R_{u_*})$, touches $S(v, 10r)$. Since the sphere $S(v, 10r)$ is a subset of $\bigcup_{u \in A(v, r, 10)} B(u, r)$ we get that this ball touches a ball of the form $B(u, r)$ for some u in $A(v, r, 10)$.

Now we shall prove that, for this u , the event $G(u, r)$ occurs. It is easy to see that we can go from $B(u, r)$ to the complement of $B(u, 8r)$ just using balls of the form $B(w, R_w)$ centered at points from $\mathcal{P} \cap B(v, 100r)$. Since $\tilde{H}(v, r)$ does not occur, the radius of any such ball is less than r . Then, we can go from $B(u, r)$ to the complement of $B(u, 8r)$ just using balls of the form $B(w, R_w)$ centered at points from $\mathcal{P} \cap B(u, 10r)$. In other words, the event $G(u, r)$ occurs. Then, the event $\bigcup_{u \in A(v, r, 10)} G(u, r)$ does occur. The proof that the event $\bigcup_{w \in A(v, r, 80)} G(w, r)$ does occur follows in the same lines. \square

The event on the right side of (12) is the intersection of two events. The first depends on what happens inside $B(v, 20r)$. The other event only depends on what happens in the region $B(v, 70r)^c$. Then, these two events are independent. It follows from (12) that

$$\begin{aligned} \mathbf{P}(G(v, 10r)) &\leq \left(\sum_{u \in A(v, r, 10)} \mathbf{P}_{p, \nu}(G(u, r)) \right) \left(\sum_{w \in A(v, r, 80)} \mathbf{P}_{p, \nu}(G(w, r)) \right) \\ &\quad + \mathbf{P}_{p, \nu}(\tilde{H}(v, r)). \end{aligned} \quad (13)$$

Notice that $|A(v, r, m)| \leq C_1 m^{\dim_A(\Gamma)}$ for all $v \in V$, $r \in \mathbb{N}$ and $m \in \mathbb{N}$. Therefore, by (13) we get

$$\sup_{v \in V} \mathbf{P}_{p, \nu}(G(v, 10r)) \leq K \left(\sup_{v \in V} \mathbf{P}_{p, \nu}(G(v, r)) \right)^2 + \sup_{v \in V} \mathbf{P}_{p, \nu}(\tilde{H}(v, r)), \quad (14)$$

where $K = C_1^2 800^{\dim_A(\Gamma)}$.

Lemma 8 *There exists positive constants C_2 and C_3 , which depends only on the Assouad dimension $\dim_A(\Gamma)$ of Γ , such that for each $v \in V$ and $r \in \mathbb{N}$, the following inequalities hold:*

$$\mathbf{P}_{p, \nu}(G(v, r)) \leq p C_2 r^{\dim_A(\Gamma)}, \quad (15)$$

$$\mathbf{P}_{p, \nu}(\tilde{H}(v, r)) \leq p C_3 \mathbf{E}_\nu [R^{\dim_A(\Gamma)} \mathbf{1}\{R \geq r\}]. \quad (16)$$

Proof of Lemma 8. Let $r \in \mathbb{N}$. A simple computation shows that

$$\begin{aligned} \mathbf{P}_{p, \nu}(G(v, r)) &\leq \mathbf{P}_{p, \nu}(\exists x \in \mathcal{P} \cap B(v, 10r)) \\ &\leq p |B(v, 10r)| \\ &\leq p C_2 r^{\dim_A(\Gamma)}, \end{aligned}$$

where $C_2 = C_1 10^{\dim_A(\Gamma)}$. In the last inequality we used inequality 3.

To show (16) we note that $\tilde{H}(v, r) = 1\{Y_v \geq 1\}$, where Y_v is a random variable defined by

$$Y_v = \sum_{u \in B(v, 100r)} 1\{u \in \mathcal{P}\} 1\{R_u \geq r\}.$$

We have

$$\begin{aligned} \mathbf{P}_{p, \nu}(\tilde{H}(v, r)) &\leq \mathbf{E}_{p, \nu}[Y_v] \\ &= \sum_{u \in B(v, 100r)} p \mathbf{P}_\nu(R_u \geq r) \\ &= p |B(v, 100r)| \mathbf{P}_\nu(R \geq r) \\ &\leq p C_3 \mathbf{E}_\nu[R^{\dim_A(\Gamma)} 1\{R \geq r\}], \end{aligned}$$

where $C_3 = C_1 100^{\dim_A(\Gamma)}$. The first equality follows from the independence between \mathcal{P} and \mathcal{R} and the second equality follows from the fact that the random variables $R_u, u \in V$ are identically distributed. \square

2.3 Proof of Theorem 1

By (10), the proof of Theorem 1 is reduced to show the existence of $p_0 > 0$ such that there exists an increasing sequence $(r_n)_{n \in \mathbb{N}}$ of natural numbers with $\lim_{n \rightarrow \infty} \mathbf{P}_{p, \nu}(G(v, r_n)) = 0$ for all $p < p_0, v \in V$. For this reason we need the following lemma.

Lemma 9 *Let f and g be two functions from \mathbb{N} to \mathbb{R}_+ satisfying the following conditions: (i) $f(r) \leq 1/2$ for all $r \in \{1, \dots, 10\}$; (ii) $g(r) \leq 1/4$ for all $r \in \mathbb{N}$; (iii) for all $r \in \mathbb{N}$:*

$$f(10r) \leq f^2(r) + g(r). \quad (17)$$

If $\lim_{r \rightarrow \infty} g(r) = 0$, then $\lim_{n \rightarrow \infty} f(10^n r) = 0$ for each $r \in \{1, \dots, 10\}$.

Proof of Lemma 9. For each $n \in \mathbb{N}$, let $F_n = \max_{1 \leq r \leq 10} f(10^n r)$ and let $G_n = \max_{1 \leq r \leq 10} g(10^n r)$. Using (17) and hypothesis (i) and (ii) we may conclude, by means of the induction principle that, for each $n \in \mathbb{N}$, $F_n \leq 1/2$ and

$$F_n \leq \frac{1}{2^{n+1}} + \sum_{j=0}^{n-1} \frac{1}{2^j} G_{n-1-j}. \quad (18)$$

Since $g(10^n r)$ goes to zero as $n \rightarrow \infty$ we have that $G_n \rightarrow 0$ when $n \rightarrow \infty$. By (18), we obtain that $F_n \rightarrow 0$ when $n \rightarrow \infty$. \square

Consider the functions $f(r) = K \sup_{v \in V} \mathbf{P}_{p, \nu}(G(v, r))$ and $g(r) = K \sup_{v \in V} \mathbf{P}_{p, \nu}(\tilde{H}(v, r))$. By (14) it follows that

$$f(10r) \leq f^2(r) + g(r). \quad (19)$$

By condition (5) and (16) we have that $\lim_{r \rightarrow \infty} g(r) = 0$ for any p .

We show that there exists $p_0 > 0$ such that if $p < p_0$ then $f(r) \leq 1/2$, $1 \leq r \leq 10$ and $g(r) \leq 1/4$, $r \in \mathbb{N}$.

Set

$$p_0 = \min \left(\frac{1}{2KC_2 10^{\dim_A(\Gamma)}}, \frac{1}{4KC_3 \mathbf{E}_\nu[R^{\dim_A(\Gamma)}]} \right).$$

By condition (5), we get $p_0 > 0$.

Let $p > 0$ be such that $p \leq p_0$. It follows from (15) that

$$f(r) \leq \frac{1}{2} \left(\frac{r}{10} \right)^{\dim_A(\Gamma)}.$$

Thus we have that if $0 < p \leq p_0$, then $\max_{1 \leq r \leq 10} f(r) \leq 1/2$.

By (16), we get

$$g(r) \leq \frac{1}{4}.$$

Finally, by Lemma 9, we have that $\lim_{n \rightarrow \infty} f(10^n r) = 0$ for each $r \in \{1, \dots, 10\}$. In particular

$$\lim_{n \rightarrow \infty} f(10^n) = \lim_{n \rightarrow \infty} K \sup_{v \in V} \mathbf{P}_{p, \nu}(G(v, 10^n)) = 0.$$

□

We finish this section by proving the complete coverage of V when there exists $v \in V$ such that $\mathbf{E}_\nu[R^{\dim_A(\Gamma)}] = \infty$.

2.4 Proof of Theorem 3

We prove that for all $r \in \mathbb{N}$, the following assertion holds:

$$\mathbf{P}_{p, \nu}(\exists w \in \mathcal{P} : B(v, r) \subset B(w, R_w)) = 1.$$

If $R_w > d(w, v) + r$, then $B(v, r) \subset B(w, R_w)$. Hence,

$$\mathbf{P}_{p, \nu}(\exists w \in \mathcal{P} : B(v, r) \subset B(w, R_w)) \geq \mathbf{P}_{p, \nu}(\exists w \in \mathcal{P} : R_w > d(w, v) + r)$$

Let $A_k(v)$ be the event defined as $A_k(v) = \{\exists w \in \mathcal{P} \cap S(v, k) : R_w > k + r\}$. It is clear that the events $A_k(v)$ are independent and that $\mathbf{P}_{p, \nu}(A_k(v)) = p|S(v, k)|\mathbf{P}_{p, \nu}(R > k + r)$. Note that

$$\begin{aligned} \sum_{k \geq 0} \mathbf{P}(A_k(v)) &= p \sum_{k \geq 0} |S(v, k)| \mathbf{P}_\nu(R > k + r) \\ &= \sum_{k \geq 0} |B(v, k)| \nu(k + r + 1). \end{aligned} \tag{20}$$

Since $\mathbf{E}_\nu[R^{\dim_A(\Gamma)}] = \infty$ we conclude that the series in the right hand side of (20) diverges. By the second Borel-Cantelli lemma we have that $\mathbf{P}(A_k(v) \text{ i.o.}) = 1$. □

3 The Number of Infinite Clusters

In this section we address the problem of determining *how many infinite connected component there can be*. We give an answer when the underlying graph has a family of symmetries which acts separating points. We begin by recalling that an isometry on a graph Γ with vertex set V is a function $g : V \rightarrow V$ preserving the geodesic distance of the graph, i.e., $d(g(v), g(w)) = d(v, w)$. We denote by $\text{Iso}(\Gamma)$ the group of isometries of Γ and we observe that isometries preserve the counting measure in Γ .

We say that a family of isometries $S \subset \text{Iso}(\Gamma)$ *acts separating points* of Γ if the orbit of any vertex of Γ by the action of S is infinite.

As a direct consequence of this definition we have also that S separates compacts (i.e., finite sets). In other words, given a compact set $K \in \Gamma$, there exists $g \in S$ such that $g(K) \cap K = \emptyset$.

Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process in Γ determined by a family of random variables $\mathcal{X} = (X_v : v \in V)$ and $\mathcal{R} = (R_v : v \in V)$. We say that an isometry $g : \Gamma \rightarrow \Gamma$ leaves the marked point process invariant if the random variables $R_{g(v)}$ and R_v are equally distributed. Henceforth we assume that the isometries $g \in S$ leave the marked point process invariant.

In order to state the result about ergodicity of the Boolean model we need to define the action of the family of isometries on the marked point process. For that purpose we assume that the marked point process is defined in the space of counting measures

$$(\hat{\Gamma}, \mathcal{A}) := (\mathcal{N}(\Gamma \times \mathbb{N}_0), \mathcal{B}(\Gamma \times \mathbb{N}_0))$$

where $\mathcal{N}(\Gamma \times \mathbb{N}_0)$ is the set of locally finite counting measures in $\Gamma \times \mathbb{N}_0$.

For each isometry $g \in S$ leaving the marked point process invariant, we induce a map $\hat{g} : (\hat{\Gamma}, \mathcal{A}, \mathbb{P}) \rightarrow (\hat{\Gamma}, \mathcal{A})$ as follows:

$$\hat{g}(\omega)(B) = \omega(g^{-1}(B)),$$

where $B \in \mathcal{B}(\Gamma \times \mathbb{N}_0)$ e $\omega \in \hat{\Gamma}$. The function \hat{g} is measurable and we observe that if g leaves the process invariant then \hat{g} is measure-preserving.

Let \mathcal{I} denote the sigma-field of events that are invariant under all isometries of S , then the measure \mathbf{P}_p is called *S-ergodic* if for each $A \in \mathcal{A}$, we have either $\mathbf{P}_p(A) = 0$ or $\mathbf{P}_p(A^c) = 0$.

We also note that the Boolean model is *insertion tolerant*, i.e., $\mathbf{P}_p(A \cup \{v\}) > 0$ for every vertex v and every measurable A determined by the marked point process $(\mathcal{X}, \mathcal{R})$ with $\mathbf{P}_p(A) > 0$. In the previous definition, we use that a vertex may be viewed as a ball of radius 0 and hence it can be identified with an element of $\mathcal{B}(\Gamma \times \mathbb{N}_0)$. The insertion tolerant property follows from a direct comparison between the discrete Boolean percolation model and the underlying Bernoulli point process. In fact, a stronger property holds:

$$\mathbf{P}_p(A \cup \{v\}) > p\mathbf{P}_p(A).$$

Clearly, the definition of insertion tolerant may be generalized to any finite set of vertices K and we may conclude that $\mathbf{P}(A \cup K) > 0$ for any finite subset K and any measurable set A satisfying $\mathbf{P}(A) > 0$. Then, an adaptation of the arguments given in [12] yields:

Theorem 10 (Ergodicity of the Boolean model) *Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process in a connected locally finite graph Γ . Assume that Γ has a family of isometries S separating points and leaving the marked point process $(\mathcal{X}, \mathcal{R})$ invariant. Then the Boolean discrete percolation model is S -ergodic.*

Proof. Let A be an S -invariant subset of $\Gamma \times \mathbb{N}_0$, i.e., a set satisfying $\hat{g}A = A$, for all $g \in S$. The idea is to show that A is almost independent of $\hat{g}A$ for some g .

Let $\epsilon > 0$. Since A is measurable, we may conclude from Theorem A.2.6.3 III in [4], that there exists a cylinder event B which depends only on some finite set K such that $\mathbf{P}_p(A \Delta B) < \epsilon$. For all $g \in S$, we have $\mathbf{P}_p(\hat{g}A \Delta \hat{g}B) = \mathbf{P}_p[\hat{g}(A \Delta B)] < \epsilon$. By assumption S acts separating points, then there exists some $g \in S$ such that K and gK are disjoint. Since gB depends only on gK , it follows that for some $g \in S$ B and gB are independent. Thus,

$$\begin{aligned} |\mathbf{P}_p(A) - \mathbf{P}_p(A)^2| &= |\mathbf{P}_p(A \cap gA) - \mathbf{P}_p(A)^2| \\ &\leq |\mathbf{P}_p(A \cap gA) - \mathbf{P}_p(B \cap gA)| + |\mathbf{P}_p(B \cap gA) - \mathbf{P}_p(B \cap gB)| \\ &\quad + |\mathbf{P}_p(B \cap gB) - \mathbf{P}_p(B)^2| + |\mathbf{P}_p(B)^2 - \mathbf{P}_p(A)^2| \\ &\leq \mathbf{P}_p(A \Delta B) + \mathbf{P}_p(gA \Delta gB) + |\mathbf{P}_p(B)\mathbf{P}_p(gB) - \mathbf{P}_p(B)^2| \\ &\quad + |\mathbf{P}_p(B) - \mathbf{P}_p(A)| (\mathbf{P}_p(B) + \mathbf{P}_p(A)) < 4\epsilon \end{aligned}$$

Therefore $\mathbf{P}_p(A) \in \{0, 1\}$. The proof is complete. \square

As a consequence of the ergodicity of the Boolean model we get the following theorem.

Theorem 11 *Let $(\mathcal{X}, \mathcal{R})$ be a Bernoulli marked point process in a connected locally finite graph Γ . Assume that Γ has a family of isometries S separating points and leaving the marked point process $(\mathcal{X}, \mathcal{R})$ invariant. Then the number of infinite clusters in the Boolean discrete percolation model is, a.s., constant and equals either 0, 1, or ∞ .*

Proof. Let N_∞ denotes the number of infinite clusters. The action of any element of S on a configuration does not change the value N_∞ . In order to prove this assertion, consider $\hat{g} \in \hat{\Gamma}$ and ω a realization of the process. Then $N_\infty(\hat{g}(\omega)(\Gamma)) = N_\infty(\omega(g^{-1}(\Gamma))) = N_\infty(\omega(\Gamma))$. In other words, $N_\infty(\hat{g}(\omega)) = N_\infty(\omega)$. Hence, N_∞ is measurable with respect to the sigma algebra of the S -invariants sets, \mathcal{I}_S . Since the Boolean model is ergodic, we may conclude that N_∞ is, a.s., constant.

Now, assume that $N_C = k \geq 2$. Let $v \in V$. Then, there exists an $R > 0$ such that $B(v, R)$ intersects all the k infinite clusters. It follows from the insertion tolerant property that the probability of the number of infinite cluster being one is positive, contradicting the hypothesis that $N_C \geq 2$ a.s. The proof is complete. \square

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